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USA Mathematical Talent Search

Year	Round	Problem
22	2	1

Let the nth row of the triangle be represented by r_n, a list containing elements of that row; $r_{i,j}$ is the jth number in the ith row of the triangle. Let the colored squares be labeled $\alpha \ldots \epsilon$ as in the given larger diagram, and the other three numbers in the top row being p, q, s such that $r_1 = [p, \alpha, \alpha, \beta, \beta, q, s]$.

By definition, two squares with the same color have the same value, so:

$$\gamma = r_{2,3} = r_{3,1}$$

$$\delta = r_{2,6} = r_{3,3}$$

$$\epsilon = r_{4,3} = r_{5,1}$$

Adding together the two elements above each, $r_2 = [\alpha + p, 2\alpha, \alpha + \beta, 2\beta, \beta + q, q + s]$, and $r_3 = [3\alpha + p, 3\alpha + \beta, \alpha + 3\beta, 3\beta + q, \beta + 2q + s]$. Looking at the definitions of γ and δ, $\alpha + \beta = 3\alpha + p$ and $q + s = \alpha + 3\beta$. Continuing to add, it can be found that $r_{4,3} = \alpha + 6\beta + q$ and $r_{5,1} = 10\alpha + 5\beta + p$, and equating those two simplifies to $q = 9\alpha - \beta + p$. We will now determine p, q, s in terms of α and β; moving terms around gives the following three equations:

$$p = \beta - 2\alpha$$

$$q = 9\alpha - \beta + p$$

$$s = \alpha + 3\beta - q$$

Substituting p into the second equation results in $q = 7\alpha$, and substituting q into the third equation results in $s = 3\beta - 6\alpha$. p, α, β, q, and s must all be one digit, so $\alpha = 1$ and $q = 7$. This simplifies p and s to $p = \beta - 2$ and $s = 3\beta - 6$. If β is a single digit (i.e. $0 < \beta < 10$), then so is p (when $\beta > 2$), but in order for s to be a single digit, then $0 < 3\beta - 6 < 10$. Solving this for β results in $2 < \beta < 6$, (the right side of the inequality, when divided by 3, was rounded up to account for the fact that β has to be an integer), so the only possible values of β are 3, 4, and 5. Below are the three triangles resulting from those choices. (β goes from 3 to 5 going left to right)

1	1	1	3	3	7	3
2	2	4	6	10	10	
4	6	10	16	20		
10	16	26	36			
26	42	62				
68	104					
172						

2	1	1	4	4	7	6
3	2	5	8	11	13	
5	7	13	19	24		
12	20	32	43			
32	52	75				
84	127					
211						

3	1	1	5	5	7	9
4	2	6	10	12	16	
6	8	16	22	28		
14	24	38	50			
38	62	88				
100	150					
250						

When $\beta = 3$, $r_{2,1} = r_{2,2}$, so $\beta \neq 3$ because all non-colored integers must be distinct. When $\beta = 4$, $r_{1,1} = r_{2,2}$, so $\beta \neq 4$ for the same reason as above. There are no such issues when $\beta = 5$, and thus the only unique solution to the problem is the boxed triangle on the right, where $(p, \alpha, \beta, q, s) = (3, 1, 5, 7, 9)$.

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USA Mathematical Talent Search

Year	Round	Problem
22	2	2

First, every tworrific sequence can be represented by the following sequence:

$$a_n = \sum_{i=1}^{n-1} 2^{k_{n-i}}(-1)^i + (-1)^n$$

where $a_0 = 1$ (by definition), and $a_{m-1} + a_m = 2^{k_m}$ where $p \geq 0$. k_n is the characteristic sequence of the particular sequence a_n; since every member a_i can be represented by the sum of powers of 2 with alternating sign, this sequence describes all tworrific sequences. k_n only has values for $n \geq 1$, as a_0 has no powers of 2 to describe.

(a) To find the minimum length, it will be assumed that the last number in any observed sequence is 2011; if not, then a shorter one with 2011 in it would simply end at that index. It is easily proven that there are no tworiffic sequences of length 2 or 3; 2012 is not a power of 2, and the only number that results in a positive sum for both 1 and 2011 that also adds up to a power of 2 when added to 2011 is 37, but 38 is not a power of 2 either.

Next, consider sequences of length 4 ($n = 0 \ldots 3$). 2011, the final number, is also equal to $2^{k_3} - 2^{k_2} + 2^{k_1} - 1$, or $2^{k_3} - 2^{k_2} + 2^{k_1} = 2012$. Dividing the equation out by 4 results in $2^{k_3-2} - 2^{k_2-2} + 2^{k_1-2} = 503$; since the left side is all even numbers unless an exponent is 0, exactly one of the exponents must be zero in order for all three to add up to an odd number, 503 (if two were, it would be an even number, and if three were, it would just be 3). If $k_1 = 2$, then $2^{k_3-2} - 2^{k_2-2} = 502$, and since no two powers of 2 have a difference of 502, $k_1 > 2$. The same goes for k_3 because it has the same sign, so k_2 would have to be 2. If $k_2 = 2$, then $2^{k_3-2} + 2^{k_1-2} = 504$, but since no two powers of 2 sum to 504, this is a contradiction; therefore, there are no tworrific sequences of length 4.

This can be done similarly to higher lengths. If $n = 0 \ldots 4$ (i.e. length 5), then $2^{k_4} - 2^{k_3} + 2^{k_2} - 2^{k_1} = 2010$. Dividing by 2 yields $2^{k_4-1} - 2^{k_3-1} + 2^{k_2-1} - 2^{k_1-1} = 1005$. If $k_1 = 1$, $2^{k_4-2} - 2^{k_3-2} + 2^{k_2-2} = 503$, which we know leads to a contradiction; likewise for $k_3 = 1$. If $k_2 = 1$, then $2^{k_4-3} - 2^{k_3-3} - 2^{k_1-3} = 251$. If we now consider $k_1 = 3$, then $2^{k_4-3} - 2^{k_3-3} = 252$, which is possible for $k_4 = 11$ and $k_3 = 5$. Therefore, the smallest possible sequence is of length $\boxed{5}$, the example of which has characteristic sequence $k_n = \{3, 1, 5, 11\}$ (that particular sequence is 1, 7, -5, 37, 2011).

(b) Because the sequences of length 5 all have the same basic structure and are symmetric in terms of parity of the index of k used (that is, k_2 and k_4 can be switched and 2011 will still be the last member of the sequence). Therefore, the only possible characteristic sequences k are permutations of $\{3, 1, 5, 11\}$ with the first and third members switched and/or the second and fourth: $\{3, 1, 5, 11\}, \{3, 11, 5, 1\}, \{5, 1, 3, 11\}$, and $\{5, 11, 3, 1\}$, for a total of $\boxed{4}$ different sequences.

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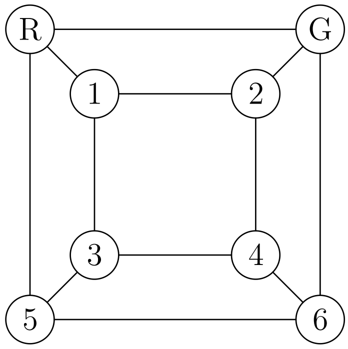
USA Mathematical Talent Search

Year	Round	Problem
22	2	3



The diagram of the cube with people standing on the vertices can be simplified to the graph on the right, where R is Richard, G is the Gortha monster, and the numbers represent the other friends. Designate the action of throwing the potato as $A \to B$ where person A throws it to person B. Then, the problem is simply finding the probability that Richard throws to the Gortha monster, or $P(R \to G)$.

In general, $P(G \to *) = 0$, where $*$ is any player, because the Gortha monster eats the potato upon catching it. Since the potato is thrown at random, the probability of a person throwing to any particular vertex is $\frac{1}{3}$ if that vertex is neighboring that person; $P(A \to B) = \frac{1}{3}$ if B neighbors A. Then 7 equations can be made relating the probability of either Richard feeding the Gortha, or another person throwing the potato to Richard (which would then be substituted into the first probability).

$$P(R \to G) = \frac{1}{3} + \frac{1}{3}P(1 \to R)P(R \to G) + \frac{1}{3}P(5 \to R)P(R \to G)$$

$$P(1 \to R) = \frac{1}{3} + \frac{1}{3}P(3 \to R) + \frac{1}{3}P(2 \to R)$$

$$P(2 \to R) = \frac{1}{3}P(4 \to R) + \frac{1}{3}P(1 \to R)$$

$$P(3 \to R) = \frac{1}{3}P(1 \to R) + \frac{1}{3}P(5 \to R) + \frac{1}{3}P(6 \to R)$$

$$P(4 \to R) = \frac{1}{3}P(3 \to R) + \frac{1}{3}P(6 \to R) + \frac{1}{3}P(2 \to R)$$

$$P(5 \to R) = \frac{1}{3} + \frac{1}{3}P(6 \to R) + \frac{1}{3}P(3 \to R)$$

$$P(6 \to R) = \frac{1}{3}P(4 \to R) + \frac{1}{3}P(5 \to R)$$

For now, we will denote $P(N \to R)$ as P_N and $P(R \to G)$ as G. The first equation can be simplified to $3G = 1 + P_1G + P_5G$, and, solving for G, results in $G = \dfrac{1}{3 - P_1 - P_5}$. Multiplying the last six equations by 3 and moving every non-probability term to the left yields new equations:

$$3P_1 - P_2 - P_3 = 1$$

$$3P_2 - P_1 - P_4 = 0$$

$$3P_3 - P_1 - P_5 - P_6 = 0$$

$$3P_4 - P_2 - P_3 - P_6 = 0$$

$$3P_5 - P_3 - P_6 = 1$$

$$3P_6 - P_4 - P_5 = 0$$

Student: Eugene Bulkin
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USA Mathematical Talent Search

Year	Round	Problem
22	2	3

If we let $\mathbf{a}$ be a square matrix with the coefficients of the six equations, $\mathbf{p}$ be the column matrix of the variables, and $\mathbf{b}$ be the column matrix of the constants on the right side, then the previous six equations are the same as $\mathbf{a} \cdot \mathbf{p} = \mathbf{b}$ where

$$\mathbf{a} = \begin{bmatrix} 3 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & 0 & 0 \\ -1 & 0 & 3 & 0 & -1 & -1 \\ 0 & -1 & -1 & 3 & 0 & -1 \\ 0 & 0 & -1 & 0 & 3 & -1 \\ 0 & 0 & 0 & -1 & -1 & 3 \end{bmatrix}, \mathbf{p} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix}, \text{and } \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

A calculator was used to calculate the inverse of the $\mathbf{a}$ (as determining the inverse of a 6 by 6 square matrix is incredibly difficult by hand):

$$\mathbf{a}^{-1} = \frac{1}{256} \begin{bmatrix} 124 & 52 & 64 & 32 & 36 & 44 \\ 57 & 123 & 48 & 56 & 31 & 45 \\ 59 & 33 & 144 & 40 & 77 & 87 \\ 47 & 61 & 80 & 136 & 57 & 91 \\ 28 & 20 & 64 & 32 & 132 & 76 \\ 25 & 27 & 48 & 56 & 63 & 141 \end{bmatrix}$$

Left ultiplying both sides of $\mathbf{a} \cdot \mathbf{p} = \mathbf{b}$ by $\mathbf{a}^{-1}$ results in $\mathbf{a}^{-1}\mathbf{b} = \begin{bmatrix} \frac{5}{8} & \frac{11}{32} & \frac{17}{32} & \frac{13}{32} & \frac{5}{8} & \frac{11}{32} \end{bmatrix}^{\mathrm{T}} = \mathbf{p}$. The first and fifth elements of this matrix $(\frac{5}{8})$ are equal to P_1 and P_5 respectively, so $G = \frac{1}{3-2\cdot\frac{5}{8}} = \boxed{\frac{4}{7}}$.

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USA Mathematical Talent Search

Year	Round	Problem
22	2	5

To solve the problem, the worst case scenario will be considered, where Zara is wrong every time she guesses (until the last time, when she is guaranteed to get it right). Without loss of generality, assume that Ada's number is always higher than Zara's (i.e. it is 2011). To guess Ada's number, Zara employs the same strategy as someone trying to guess a fixed number, except the numbers are slightly different.

First, Zara guesses 1006, as this provides the most even distribution of wrong numbers (the sets of integers 1 through 1005 and 1007 through 2011 are the same length). When Ada says her number is higher, now the possible range decreases from $[1, 2011]$ to $[1006, 2010]$, because Ada's number changes by ± 1, but if her number is at 2011, she must go down. Then, Zara guesses the number in between those, and Ada changes hers, and so on.

In general, if we consider the numbers $1 \ldots 2m+1$, a recurrence relation can be made that determines what number Zara will guess; in this case, $a_0 = 1$ (no guess yet), $a_1 = m+1$, and so on. The range of numbers that Zara has to look at is from her previous guess (a_{n-1}) to either $2m$ or $2m+1$ (if Ada's number was $2m+1$, she would have to subtract one, but if it was $2m$ it now can go back up). The general form for this relation is $a_n = m + \frac{1}{2}a_{n-1} + \frac{1}{4}(1-(-1)^n)$. Using Mathematica's RSolve function (which solves recurrence relations), a closed form is found:

$$a_n = \frac{1}{6}(-2^{-n}(12m + (-2)^n - 4) + 12m + 3)$$

. As $n \to \infty$, $a_n \to 2m + \frac{1}{2}$; since Zara will take the floor of each term of the sequence (she can't guess a non-integer, but taking the floor of each term will not affect when the sequence reaches its limit), the sequence is guaranteed to surpass $2m$, and thus when she takes the floor, it will be exactly $2m$. This is sufficient for guessing Ada's number, since if Ada's number is larger than $2m$, it is $2m+1$, and thus the only possible number Ada can move to when Zara guesses $2m$ is $2m$ as well, and Zara will then guess $2m$ again to win.

There is no clean general solution for the first n such that $a_n \geq 2m$, but one can be found for when $m = 1005$. Substituting 1005 for m in a_n results in $a_n = \frac{1}{6}(-(-1)^n - 15072^{3-n} + 12063)$. All that is necessary is to find the first n such that $a_n \geq 2010$. This is a non-algebraic equation, but using a calculator finds that $a_n = 2010$ at $n \approx 12.3$, and rounding up results in the minimum number of guesses, $\boxed{13}$.

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USA Mathematical Talent Search

Year	Round	Problem
22	2	6

For ease of reading, designate a movement of the robot from number a to number b, using n as the selected integer, as $a \to b$.

Lemma 1. *For all positive integers $x \geq 2$ there exists a positive integer n such that $x \to \frac{1}{x-1}$.*

Proof. $x \to \dfrac{1}{x-1}$ implies $\dfrac{x+n}{1+xn} = \dfrac{1}{x-1}$. Cross-multiplying results in $1 + xn = (x+n)(x-1) = x^2 + xn - x - n$, or $n = x^2 - x - 1$. n will always be an integer if x is an integer, and for n to be positive, $x^2 - x - 1 > 0$. The roots of this quadratic are $\approx \pm 1.618$. The smallest positive integer such that n is a positive integer is 2, so $x \geq 2$. $\square$

Lemma 2. *For all positive integers $x \geq 2$ there exists a positive integer n such that $\frac{1}{x} \to x - 1$.*

Proof. $\dfrac{1}{x} \to x - 1$ implies $\dfrac{1+xn}{x+n} = x - 1$, which, after cross-multiplying, results in the same equation above, $1 + xn = (x+n)(x-1)$, and thus the same result follows. $\square$

Lemma 3. *For all odd positive integers $x \geq 3$ there exists a positive integer n such that $x \to \frac{1}{x-2}$.*

Proof. $x \to \dfrac{1}{x-2}$ implies $\dfrac{x+n}{1+xn} = \dfrac{1}{x-2}$, and after cross-multiplying this yields $1 + xn = (x + n)(x-2) = x^2 + xn - 2x - 2n$, or $n = \frac{1}{2}(x^2 - 2x - 1)$. Because $x^2 - 1$ is even for all odd x, n will be an integer for only odd x. The positive root of this quadratic is $x = 1 + \sqrt{2} \approx 2.414$, so the only possible positive integer values for x are those ≥ 3. $\square$

If the robot begins at 2011, then using lemma 3, there exists a valid n such that $2011 \to \frac{1}{2009}$. It can then move from $\frac{1}{2009}$ to 2008 using lemma 2, and so on, such that the robot moves along the pattern $2011 \to \frac{1}{2009} \to 2008 \to \ldots \to 2$.

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USA Mathematical Talent Search

Year	Round	Problem
22	2	6

Note: Of course, this is nowhere near an optimal solution. I conjecture that the shortest possible length of a sequence is six. One possible solution is $2011 \to \frac{1}{671} \to 111 \to \frac{1}{23} \to 7 \to \frac{1}{3} \to 2$. While I was unable to find a proof for the fact that the robot cannot travel from 2011 to 2 in four steps, I did prove that it cannot do it in two, shown below.

Lemma 4. *The robot oscillates between $(0, 1)$ and $(1, \infty)$, alternating each step.*

Proof. let $x = \frac{p}{q} < 1$, $q > p$ and assume $\frac{p+nq}{q+np} < 1$. This implies $q+np > p+nq$ or $q-p > n(q-p)$, but n must be a positive integer > 1, so this is a contradiction. If $x = \frac{p}{q} > 1$, $p > q$, and the same contradiction occurs. Thus, $x < 1 \leftrightarrow x \to y$, $y > 1$, and $x > 1 \leftrightarrow x \to y$, $y < 1$. $\square$

Theorem 1. *There are no sequences of steps of odd length from one integer to another.*

Proof. Using lemma 4, it is clear that the robot, when at an integer, must go through a number less than 1 before it can reach another integer; that is, for the robot to go from $(1, \infty) \to (1, \infty)$, first it must go from $(1, \infty) \to (0, 1)$. Thus, to go from one integer to another, a robot must make at least two steps, and this proceeds for any integer because $\mathbb{Z}^+ - \{0, 1\} \subset (1, \infty)$, where $\mathbb{Z}^+$ is the set of non-negative integers. $\square$

Lemma 5. *All $x = \frac{p}{q}$, $p < q$ such that $x \to 2$ are of the form $x = \frac{k}{2k+3}$.*

Proof. In order for $\frac{p}{q} \to 2$, $\frac{p+qn}{q+pn} = 2$, or $n = \frac{p-2q}{2p-q}$. In order for n to be an integer, $\frac{p-2q}{2p-q}$ must also be an integer. Using long division, $\frac{p-2q}{2p-q} = \pm\frac{3p}{q-2p} + 2$ (sign depending on whether $x \in (0, \frac{1}{2})$ or $(\frac{1}{2}, 1)$); therefore, for n to be an integer, $\frac{3p}{q-2p} = k$ for some integer k. Solving for $\frac{p}{q}$ results in $\frac{p}{q} = \frac{k}{2k+3}$, which makes $n = k + 2$, an integer, so $x = \frac{k}{2k+3}$ always has an integer n such that $x \to 2$. $\square$

Theorem 2. *There exist no sequences of length 2 from 2011 to 2.*

Proof. Using lemma 5, in order for 2011 to reach 2 in two steps, $2011 \to \frac{k}{2k+3}$ must be possible for some integers n, k. Solving the equation $\frac{n+2011}{2011n+1} = \frac{k}{2k+3}$ for n results in $n = \frac{4021k+6033}{2009k-3}$, or, after division, $n = \frac{1}{2009}(\frac{12132360}{2009k-3} + 4021)$. For n to be an integer, $2009k - 3$ must divide 12132360 and $\frac{12132360}{2009k-3} + 4021$ must be a multiple of 2009. 2009 can be factored as $7^2 \cdot 41$, and 12132360 to $2^3 \cdot 3^2 \cdot 5 \cdot 67 \cdot 503$, which makes 96 divisors. Adding 3 to each divisor and dividing by 49 results in the only integers being 539 and 22638, but dividing by 41 results only results in 123 and 101106. Since no divisor of 12132360 plus 3 shares the factors of 2009, there is no $k > 0$ such that n will ever be an integer, making $2011 \to \frac{k}{2k+3}$ impossible. $\square$